B84

3 次方程式$x^3+px^2+qx+r=0$の異なる 3 つの解をα,β,γとおく。このとき，

$A=\dfrac{a_3-(\beta+\gamma)a_2+\beta\gamma a_1}{(\alpha-\beta)(\alpha-\gamma)},B=\dfrac{a_3-(\gamma+\alpha)a_2+\gamma\alpha a_1}{(\beta-\gamma)(\beta-\alpha)},C=\dfrac{a_3-(\alpha+\beta)a_2+\alpha\beta a_1}{(\gamma-\alpha)(\gamma-\beta)}$とおくと，

$a_n=A\alpha^{n-1}+B\beta^{n-1}+C\gamma^{n-1}$は，$a_{n+3}+pa_{n+2}+qa_{n+1}+ra_n=0\quad(n=1,2,3,\cdots)$を満たすことを

数学的帰納法で証明せよ。

（証明）$a_n=A\alpha^{n-1}+B\beta^{n-1}+C\gamma^{n-1}\cdots$①とおく。

[1]　$n=1$のとき，

（①の右辺）

$$=A+B+C=\frac{a_3-(\beta+\gamma)a_2+\beta\gamma a_1}{(\alpha-\beta)(\alpha-\gamma)}+\frac{a_3-(\gamma+\alpha)a_2+\gamma\alpha a_1}{(\beta-\gamma)(\beta-\alpha)}+\frac{a_3-(\alpha+\beta)a_2+\alpha\beta a_1}{(\gamma-\alpha)(\gamma-\beta)}$$

$$=\frac{-(\beta-\gamma)\{a_3-(\beta+\gamma)a_2+\beta\gamma a_1\}-(\gamma-\alpha)\{a_3-(\gamma+\alpha)a_2+\gamma\alpha a_1\}-(\alpha-\beta)\{a_3-(\alpha+\beta)a_2+\alpha\beta a_1\}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

=★

（分子のa_3の係数）$=-(\beta-\gamma)-(\gamma-\alpha)-(\alpha-\beta)=0$

（分子のa_2の係数）$=(\beta^2-\gamma^2)+(\gamma^2-\alpha^2)+(\alpha^2-\beta^2)=0$

（分子のa_1の係数）$=-(\beta-\gamma)\beta\gamma-(\gamma-\alpha)\gamma\alpha-(\alpha-\beta)\alpha\beta=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

★$=a_1=$（①の左辺）より，①は成り立つ。

同様に，$n=2$のとき，

（①の右辺）

$$=\alpha A+\beta B+\gamma C$$

$$=\alpha\cdot\frac{a_3-(\beta+\gamma)a_2+\beta\gamma a_1}{(\alpha-\beta)(\alpha-\gamma)}+\beta\cdot\frac{a_3-(\gamma+\alpha)a_2+\gamma\alpha a_1}{(\beta-\gamma)(\beta-\alpha)}+\gamma\cdot\frac{a_3-(\alpha+\beta)a_2+\alpha\beta a_1}{(\gamma-\alpha)(\gamma-\beta)}$$

$$=\frac{-\alpha(\beta-\gamma)\{a_3-(\beta+\gamma)a_2+\beta\gamma a_1\}-\beta(\gamma-\alpha)\{a_3-(\gamma+\alpha)a_2+\gamma\alpha a_1\}-\gamma(\alpha-\beta)\{a_3-(\alpha+\beta)a_2+\alpha\beta a_1\}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

$=a_2$

＝（①の左辺）より，①は成り立つ。

同様に，$n=3$のとき，

（①の右辺）

$$=\alpha^2A+\beta^2B+\gamma^2C$$

$$=\alpha^2\cdot\frac{a_3-(\beta+\gamma)a_2+\beta\gamma a_1}{(\alpha-\beta)(\alpha-\gamma)}+\beta^2\cdot\frac{a_3-(\gamma+\alpha)a_2+\gamma\alpha a_1}{(\beta-\gamma)(\beta-\alpha)}+\gamma^2\cdot\frac{a_3-(\alpha+\beta)a_2+\alpha\beta a_1}{(\gamma-\alpha)(\gamma-\beta)}$$

$$=\frac{-\alpha^2(\beta-\gamma)\{a_3-(\beta+\gamma)a_2+\beta\gamma a_1\}-\beta^2(\gamma-\alpha)\{a_3-(\gamma+\alpha)a_2+\gamma\alpha a_1\}-\gamma^2(\alpha-\beta)\{a_3-(\alpha+\beta)a_2+\alpha\beta a_1\}}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}$$

$=a_3$

＝（①の左辺）より，①は成り立つ。

[2]　$n=k,k+1,k+2$のとき，①が成り立つと仮定する。

すなわち，$a_k=A\alpha^{k-1}+B\beta^{k-1}+C\gamma^{k-1},a_{k+1}=A\alpha^k+B\beta^k+C\gamma^k,a_{k+2}=A\alpha^{k+1}+B\beta^{k+1}+C\gamma^{k+1}\cdots$②

$n=k+3$のとき，与えられた漸化式から

$$
\begin{aligned}
&a_{k+3}=-pa_{k+2}-qa_{k+1}-ra_k\\
&=-p\left(A\alpha^{k+1}+B\beta^{k+1}+C\gamma^{k+1}\right)-q\left(A\alpha^{k}+B\beta^{k}+C\gamma^{k}\right)-r\left(A\alpha^{k-1}+B\beta^{k-1}+C\gamma^{k-1}\right) \quad \because ②より\\
&=A\left(-p\alpha^2-q\alpha-r\right)\alpha^{k-1}+B\left(-p\beta^2-q\beta-r\right)\beta^{k-1}+C\left(-p\gamma^2-q\gamma-r\right)\beta^{k-1}\\
&=A\alpha^3\alpha^{k-1}+B\beta^3\beta^{k-1}+C\gamma^3\gamma^{k-1} \quad \because \alpha,\beta,\gamma は特性方程式の解であるから\\
&=A\alpha^{k+2}+B\beta^{k+2}+C\gamma^{k+2}
\end{aligned}
$$

これは$n=k+3$のときにも①が成り立つことを示している。

以上，[1]，[2]から数学的帰納法により，すべての自然数nに対して①は成り立つ。■