いろいろな数列の和

【基本公式】

1. $\sum_{k=1}^{n}\{a+(k-1)d\}=a+(a+d)+(a+2d)+\cdots+\{a+(n-1)d\}=\frac{1}{2}n\{2a+(n-1)d\}$ （等差数列の和）

2. $\sum_{k=1}^{n}ar^{k-1}=a+ar+ar^2+\cdots+ar^{n-1}=\frac{a(1-r^n)}{1-r}=\frac{a(r^n-1)}{r-1}$ $[r\neq 1]$ （等比数列の和）

3. $\sum_{k=1}^{n}k=\frac{1}{2}\sum_{k=1}^{n}\{k(k+1)-(k-1)k\}=\frac{1}{2}n(n+1)$

4. $\sum_{k=1}^{n}k^2=\frac{1}{6}\sum_{k=1}^{n}\{k(k+1)(2k+1)-(k-1)k(2k-1)\}=\frac{1}{6}n(n+1)(2n+1)$

5. $\sum_{k=1}^{n}k^3=\frac{1}{4}\sum_{k=1}^{n}\{k^2(k+1)^2-(k-1)^2k^2\}=\frac{1}{4}n^2(n+1)^2$

【パターン 1】

1. $\sum_{k=1}^{n}k=\frac{1}{2}\sum_{k=1}^{n}\{k(k+1)-(k-1)k\}=\frac{1}{2}n(n+1)$

2. $\sum_{k=1}^{n}k(k+1)=\frac{1}{3}\sum_{k=1}^{n}\{k(k+1)(k+2)-(k-1)k(k+1)\}=\frac{1}{3}n(n+1)(n+2)$

3. $\sum_{k=1}^{n}k(k+1)(k+2)=\frac{1}{4}\sum_{k=1}^{n}\{k(k+1)(k+2)(k+3)-(k-1)k(k+1)(k+2)\}=\frac{1}{4}n(n+1)(n+2)(n+3)$

4. $\sum_{k=1}^{n}k(k+1)(k+2)(k+3)=\frac{1}{5}\sum_{k=1}^{n}\{k(k+1)(k+2)(k+3)(k+4)-(k-1)k(k+1)(k+2)(k+3)\}=\frac{1}{5}n(n+1)(n+2)(n+3)(n+4)$

5. $\sum_{k=1}^{n}k(k+1)(k+2)(k+3)(k+4)=\frac{1}{6}\sum_{k=1}^{n}\{k(k+1)(k+2)(k+3)(k+4)(k+5)-(k-1)k(k+1)(k+2)(k+3)(k+4)\}$

$=\frac{1}{6}n(n+1)(n+2)(n+3)(n+4)(n+5)$

【パターン 2】

1. $\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1-\frac{1}{n+1}=\frac{n}{n+1}$

2. $\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{2}\sum_{k=1}^{n}\left\{\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\right\}=\frac{1}{2}\left\{\frac{1}{1\cdot 2}-\frac{1}{(n+1)(n+2)}\right\}=\frac{n(n+3)}{4(n+1)(n+2)}$

3. $\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)(k+3)}=\frac{1}{3}\sum_{k=1}^{n}\left\{\frac{1}{k(k+1)(k+2)}-\frac{1}{(k+1)(k+2)(k+3)}\right\}=\frac{1}{3}\left\{\frac{1}{1\cdot 2\cdot 3}-\frac{1}{(n+1)(n+2)(n+3)}\right\}$

$=\frac{n(n^2+6n+11)}{18(n+1)(n+2)(n+3)}$

【パターン 2 の類型】

1. $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{n}{2n+1}$

2. $\sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)(3k+5)}=\frac{1}{6}\sum_{k=1}^{n}\left\{\frac{1}{(3k-1)(3k+2)}-\frac{1}{(3k+2)(3k+5)}\right\}=\frac{1}{6}\left\{\frac{1}{2\cdot 5}-\frac{1}{(3n+2)(3n+5)}\right\}=\frac{n(3n+7)}{20(3n+2)(3n+5)}$

【パターン 3】階乗を含む数列の和

1. $\sum_{k=1}^{n} k!k = \sum_{k=1}^{n} k!\{(k+1)-1\} = \sum_{k=1}^{n}\{(k+1)!-k!\} = (n+1)!-1$

2. $\sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{n} \frac{k+1-1}{(k+1)!} = \sum_{k=1}^{n}\left\{\frac{1}{k!}-\frac{1}{(k+1)!}\right\} = 1-\frac{1}{(n+1)!}$

【パターン 4】 $S_r = \sum_{k=1}^{n} k^r x^{k-1} = 1+2^r x+3^3 x^2+\cdots+n^r x^{n-1}$

1. $S_1 = 1+2x+3x^2+\cdots+nx^{n-1} = \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$

2. $S_2 = 1^2+2^2x+3^2x^2+\cdots+n^2x^{n-1} = \frac{n^2x^{n+2}-(2n^2+2n-1)x^{n+1}+(n+1)^2x^n-x-1}{(x-1)^3}$

3. $S_3 = 1^3+2^3x+3^3x^2+\cdots+n^3x^{n-1} = \frac{n^3x^{n+3}-(3n^3+3n^2-3n+1)x^{n+2}+(3n^3+6n^2-4)x^{n+1}-(n+1)^3x^n+x^2+4x+1}{(x-1)^4}$

4. $S_4 = 1^4+2^4x+3^4x^2+\cdots+n^4x^{n-1} = \{n^4x^{n+4}-(4n^4+4n^3-6n^2+4n-1)x^{n+3}+(6n^4+12n^3-6n^2-12n+11)x^{n+2}$
$-(4n^4+12n^3+6n^2-12n-11)x^{n+1}+(n+1)^4x^n-x^3-11x^2-11x-11\}/(x-1)^5$

【応用】

1. $r \neq 1$ のとき， $\sum_{k=1}^{n}\{a+(k-1)d\}r^{k-1} = a+(a+d)r+(a+2d)r^2+\cdots+\{a+(n-1)d\}r^{n-1}$

$= \frac{a-(a-d)r-(a+nd)r^n+\{a+(n-1)d\}r^{n+1}}{(1-r)^2} = \left(\frac{d}{r-1}n-c\right)r^n+c$　ただし， $c = \frac{a-(a-d)r}{(r-1)^2}$

2. $\sum_{k=1}^{n}\frac{5k}{k^4+4} = \frac{5}{4}\sum_{k=1}^{n}\left(\frac{1}{k^2-2k+2}-\frac{1}{k^2+2k+2}\right) = \frac{5}{4}\sum_{k=1}^{n}\left\{\frac{1}{(k-1)^2+1}-\frac{1}{(k+1)^2+1}\right\}$

$= \frac{5}{4}\left\{\frac{1}{1}+\frac{1}{1^2+1}-\frac{1}{n^2+1}-\frac{1}{(n+1)^2+1}\right\} = \frac{5}{4}\left(\frac{3}{2}-\frac{1}{n^2+1}-\frac{1}{n^2+2n+2}\right) = \frac{5n(n+1)(3n_2+3n+2)}{8(n^2+1)(n^2+2n+2)}$

3. $\sum_{k=1}^{n}\frac{2k+1}{(k^2+k+1)^2+1} = \sum_{k=1}^{n}\frac{2k+1}{(k^2+1)(k^2+2k+2)} = \sum_{k=1}^{n}\left\{\frac{1}{k^2+1}-\frac{1}{(k+1)^2+1}\right\} = \frac{1}{1^2+1}-\frac{1}{(n+1)^2+1}$

$= \frac{n(n+2)}{2(n^2+2n+2)}$

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